[11:00] Recap: FIR filters and convolution

For an FIR filter, the coefficients b_k are synonymous with the impulse response $h[n]$.

The output of an LTI system is obtained by convolving the input with the IR.

Let $x[n] = \delta[n]$. Then, the output $h[n]$ is

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$
$$= \begin{cases} b_n & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[n]x[n-k] \\ &= \underbrace{\sum_{k=0}^{M} b_k x[n-k]}_{\substack{\text{for an FIR filter with} \\ M-1 \text{ coefficients}}} \end{aligned}$$

Deconvolution can be used to obtain $x[n]$ from $y[n]$ and $h[n]$ or to obtain $h[n]$ from $y[n]$ and $x[n]$.

[11:15-11:40] Linear time invariant systems

Time invariance. Let $x[n]$ be an arbitrary input to a system $\mathcal{T}$ and let $y[n]$ be the corresponding output. The system $\mathcal{T}$ is time-invariant if, for any time shift n_0.

$$\mathcal{T}\{x[n-n_0]\} = y[n-n_0]$$

Example: $y[n] = x^2[n]$.

$$\mathcal{T}\{x[n-n_0]\} = (x[n-n_0])^2 = x^2[n-n_0] = y[n-n_0] \qquad \text{(Yes, this is TI)}$$

Additivity. Let $x_1[n]$ and $x_2[n]$ be arbitrary input signals to a system $\mathcal{T}$. The system satisfies additivity if

$$\mathcal{T}\{x_1[n] + x_2[n]\} = \mathcal{T}\{x_1[n]\} + \mathcal{T}\{x_2[n]\}$$

Example: $y[n] = x^2[n]$

$$\begin{aligned} \mathcal{T}\{x_1[n] + x_2[n]\} &= x_1^2[n] + 2x_1[n]x_2[n] + x_2^2[n] \\ &\ne y_1[n] + y_2[n] \end{aligned} \qquad \text{(No, does not satisfy additivity)}$$

Homogeneity. Let $x[n]$ be an arbitrary input to a system $\mathcal{T}$. The system satisfies homogeneity if, for any constant a,

$$\mathcal{T}\{a \cdot x[n]\} = a \cdot \mathcal{T}\{x[n]\}$$

Example: $y[n] = x^2[n]$

$$\mathcal{T}\{ax[n]\} = a^2x^2[n] \ne ax^2[n] = y^2[n] \qquad \text{(no, does not satisfy homogeneity)}$$

Linear time-invariant. A system is linear time-invariant (LTI) if it satisfies additivity, homogeneity, and time-invariance properties. A common way for a system to violate these properties is if the system has nonzero initial conditions.

[11:50-] Role of initial conditions

Example: unit delay $y[n] = \mathcal{T}\{x[n]\} = x[n-1]$

The first input value is $x[0]$ and the first output value is $y[0]$.

If we observe for $-\infty < n < \infty$, we can always compute the output using

$$y[n] = x[n-1]$$

If we only observe for $n \geq 0$, we cannot determine $y[0]$ from the input. Instead, $y[0]$ is the initial value in the memory of the ideal delay block.

$$y[0] = ?$$

$$y[1] = x[0]$$

$$y[2] = x[1]$$

$$\vdots = \vdots$$

To satisfy homogeneity, we require that $\mathcal{T}\{a \cdot x[n]\} = a \cdot \mathcal{T}\{x[n]\}$

All-zero input test (check if $\mathcal{T}\{0\} = 0$)

$$x[0] = 0, \qquad x[1] = 0, \qquad x[2] = 0, \qquad \ldots$$

$$y[0] = \text{IC}, \qquad y[1] = 0, \qquad y[2] = 0, \qquad \ldots$$

Thus for the unit delay, homogeneity is only satisfied when the initial condition is zero.

[12:05-] Linear constant coefficient differential equation

$$y(t) = a_1 y'(t) + \cdots = x(t) + b_1 x'(t) + \cdots, \qquad \text{for } t \geq 0$$

$$y(t) = \underbrace{h(t) * x(t)}_{\substack{\text{response due to zero} \\ \text{initial conditions}}} + \underbrace{y(0)}_{\substack{\text{response due to} \\ \text{nonzero initial conditions}}}$$

Doc cam

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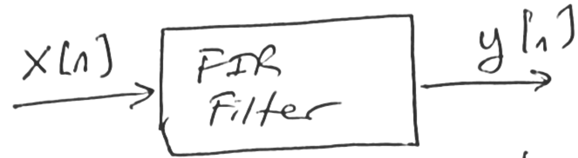
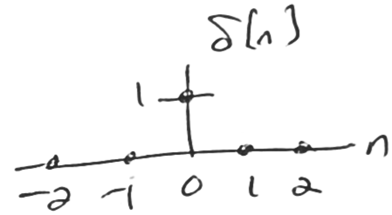
Review

Convolution



M+1 FIR filter coefficients b_k

$$y[n] = \sum_{k=0}^{M} b_k \, x[n-k]$$



Let input $x[n] = \delta[n]$,

output $h[n]$ is

$$h[n] = \sum_{k=0}^{M} b_k \, \delta[n-k]$$

↑ has value of one at $n=0$.

$$h[n] = \begin{cases} b_n & \text{for } 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

More generally, for linear time-invariant systems, $y[n] = h[n] * x[n]$ convolution

$$= \sum_{k=0}^{M} h[k] \, x[n-k]$$

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Additivity

$$x_1[n] + x_2[n] \longrightarrow \boxed{(\cdot)^2} \longrightarrow y_{additive}[n] = \cancel{y[n]} = (x_1[n] + x_2[n])^2$$

$$y_{additive}[n] = \cancel{y[n]} = \underbrace{x_1^2[n]}_{\checkmark} + \underbrace{2x_1[n]x_2[n]}_{\times \text{ doesn't work}} + \underbrace{x_2^2[n]}_{\checkmark}$$

$$y_{additive}[n] \stackrel{?}{=} y_1[n] + y_2[n]$$
$$= x_1^2[n] + x_2^2[n] \quad \text{NO}$$

Not additive.

Homogeneity.

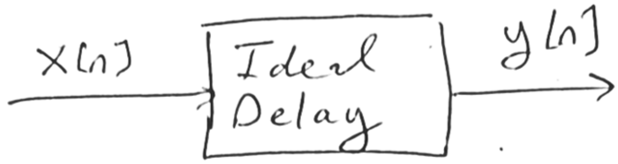
$$\alpha x[n] \longrightarrow \boxed{(\cdot)^2} \longrightarrow y_{scaled}[n] = (\alpha x[n])^2$$

$$y_{scaled}[n] = \alpha y[n] \quad \text{for all } \alpha$$
$$\alpha^2 x^2[n] = \alpha x^2[n] \quad \text{only for } \alpha = 0 \text{ \& } \alpha = 1$$

Not homogeneous.

Alternative Approach

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{} \longrightarrow y_{combined}[n] \stackrel{?}{=} \alpha y_1[n] + \beta y_2[n]$$



If we observe for $-\infty < n < \infty$,

It's LTI

$$y[n] = x[n-1]$$

If we observe for $n \geq 0$,

$$y[0] = ?$$

Initial value in the memory of the ideal delay block.

$$y[1] = x[0]$$
$$y[2] = x[1]$$
$$\vdots \qquad \vdots$$

Homogeneity. Input $\alpha\, x[n] \rightarrow y_{scaled}[n]$

For the property to hold, $y_{scaled}[n] = \alpha\, y[n]$.

All-zero input test for $\alpha = 0$:

Input is zero for $n \geq 0$

Output is $y[0], y[1], y[2], \ldots$

Initial condition must be zero $\rightarrow$ IC, 0, 0, ...

Linear Constant Coefficient
? Differential Equation

$$y(t) + a_1 y'(t) + \cdots = x(t) + b_1 x'(t) + \cdots$$

for $t \geq 0$

$$y(t) = \underbrace{h(t) * x(t)}_{\text{response due to zero initial conditions}} + \underbrace{\quad\quad}_{\text{response due to non-zero initial conditions and } x(t)=0}$$

Initial conditions

$$\begin{bmatrix} y(0) \\ y'(0) \\ \vdots \end{bmatrix} \quad \begin{bmatrix} x(0) \\ x'(0) \\ \vdots \end{bmatrix} \leftarrow ?$$

state of the system